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ON THE ANALYSIS OF ANISOTROPIC RECTANGULAR PLATES

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FOREWORD

This report was prepared by the Plastics and Composites Branch, Nonmetallic Materials Division, Air Force Materials Laboratory. The work was initiated under Project 7340, "Nonmetallic and Composite Materials," Task 734003, "Structural Plastics and Composites," and was administered under the direction of the Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio; Dr. J. M. Whitney of the Plastics and Composites Branch (LNC) was the Project Engineer.

The report covers work conducted January 1971 through March 1972.

This technical report has been reviewed and is approved.

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ABSTRACT

Extensive use of energy methods in conjunction with classical beam mode functions has been used to obtain approximate solutions to homogeneous, or symmetrically laminated, anisotropic plate problems. Because of the existence of cross-elasticity bending stiffness terms, the beam functions do not satisfy the natural boundary conditions. As a result, bending moments and stresses, which are of practical interest, may converge to the wrong solution or may not converge at all. Furthermore, bending deflections, buckling loads, and fundamental vibration frequencies converge very slowly for highly anisotropic materials. This report shows that improved results can be obtained for anisotropic plates which contain strong cross-elasticity effects by using a classical Fourier analysis which satisfies both the geometric and natural boundary conditions. Numerical results are presented for bending under transverse load, buckling under biaxial compression and pure shear, and natural frequencies of flexural vibration. Both homogeneous and laminated plates are discussed.

NOTATION

a, b	Plate length and width
D_{ij}	Elements of anisotropic bending stiffness matrix, $Q_{ij} h^3/12$ for homogeneous plates
h	Plate thickness
h_k	Distance from plate mid-plane to interface between the k and k + 1 layers of a laminate
L	Number of layers in a laminated plate
M_x, M_y	Distributed bending moments
M_{xy}	Distributed twisting moments
N_x, N_y	Distributed normal stress resultants
N_{xy}	Distributed shear stress resultant
P	Integral of the density through the plate thickness
q	Distributed normal load over plate surface
Q_{ij}	Elements of the anisotropic reduced stiffness matrix
Q'_{ij}	Reduced stiffnesses rleative to x', y' axes
t	Time
w	Plate deflection
ω	Circular frequency
σ_x, σ_y	Normal stress
σ_{xy}	Shear stress
θ	Angle of fiber axis, x'. relative to x axis of plate

INTRODUCTION

With the development of high performance fiber reinforced composite materials for structural applications has come an increased interest in solutions to anisotropic plate problems. A large number of solutions exist for bending, buckling, and free vibration of specially orthotropic rectangular plates in which the principal elastic axes are parallel to the sides of the plate. Many of these solutions are summarized in References [1] - [3]. In most structural applications, however, fiber reinforced composites are constructed of unidirectional plies in which the fiber axis is oriented at an angle θ to the x axis as illustrated in Figure 1. For such a composite symmetrically laminated about the mid-plane, the bending response is governed by the flexural equation of a homogeneous anisotropic plate [4,5], including the cross-elasticity bending stiffness terms D_{16} and D_{26}.

Energy methods have been used [6,7,8,9,10] in conjunction with classical beam mode functions to obtain approximate solutions for bending, buckling, and vibration of homogeneous, or symmetrically laminated, anisotropic plates with various boundary conditions. Extensive numerical results for rectangular plates appear in Reference [3]. Because of the existence of cross-elasticity bending stiffness terms, however, the beam functions do not satisfy the natural boundary conditions. Recent work by Fraser and Miller [11] involved the use of a generalized Ritz method in conjunction with a Fourier series and Lagrange multiplier technique to obtain an upper and lower bound on buckling loads for homogeneous anisotropic plates. Again, as in the previously cited literature, the natural boundary conditions were not satisfied.

Ashton [10] has shown that unless the natural boundary conditions are satisfied, bending moments, shear resultants, and edge reactions, which are of particular interest in bending problems, may converge to the wrong solution or may not converge at all. Furthermore, for highly anisotropic materials bending deflections, buckling loads, and free vibration frequencies converge very slowly.

It is the purpose of the present paper to show that improved results compared to existing energy solutions can be obtained for anisotropic plates having strong cross-elasticity effects by using a classical Fourier analysis which satisfies both the geometric and natural boundary conditions. An exact solution to the governing equations is obtained in the form of a double sine series plus a double cosine series. A procedure similar to that used by Green [12] and by Fletcher and Thorne [13] on isotropic plates is employed to exactly satisfy both the geometric and natural boundary conditions. Although the Fourier method is applicable to any of the classical boundary conditions, the present work considers plates which are either simply-supported on all edges, or simply-supported on two opposite edges with the adjacent edges clamped. These boundary conditions are sufficient to show improved results compared to existing energy solutions. The Fourier approach has been recently applied to anisotropic rectangular plates having all edges clamped [14]. Since all of the proper boundary conditions were geometric in nature, excellent agreement with existing Ritz solutions was obtained. This procedure has also been applied to unsymmetrically laminated anisotropic plates. [15, 16]. The laminating sequence, however, was such that the D_{16} and D_{26} cross-elasticity bending stiffness terms vanished. This simplification lead to displacement and stress function solutions which could be expressed in terms of a single function in the form of a double trigometric series.

ANALYSIS

For the purpose of completeness it is appropriate to briefly review the Fourier method of solution as applied to the anisotropic plate equation. Denoting partial differentiation by a comma, the governing equation for the bending and flexural vibration of a homogeneous, or symmetrically laminated, plate with pre-buckling inplane loads included is of the form [1, 2, 3]

$$
\begin{aligned}
& D_{11}\, w,_{xxxx} + 4D_{16}\, w,_{xxxy} + 2(D_{12} + 2D_{66})\, w,_{xyyy} \\
& + 4D_{26}\, w,_{xyyy} + D_{22}\, w,_{yyyy} + P\, w,_{tt} \qquad (1) \\
& = N_x\, w,_{xx} + 2N_{xy}\, w,_{xy} + N_y\, w,_{yy}
\end{aligned}
$$

For laminates symmetrically oriented about their mid-plane [3, 4, 5]

$$
D_{ij} = 2 \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)} z^2 dz \qquad (2)
$$

where K = L/2 for a laminate having an even number of layers and (L + 1) /2 for a laminate having an odd number of layers.

Bending Under Transverse Load

Consider a rectangular plate which lies in the region $0 \leq x \leq a$, $0 \leq y \leq b$ (Fig. 1) and subjected to a general transverse load q which can be presented by the Fourier series

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{3}$$

$$(0 < x < a, \; 0 < y < b)$$

A solution to eq. (1), in the absence of inplane forces and inertia, of the form

$$w(x, y) = w_1(x, y) + w_2(x, y) \tag{4}$$

is sought in the region $0 < x < a$, $0 < y < b$ where w_1 and w_2 independently satisfy the geometric boundary conditions. It is assumed that

$$\begin{aligned} w_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ w_2 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \tag{5}$$

Differentiating w_1 term-by-term with respect to x twice yields

$$w_{1,xx} = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \pi^2}{a^2} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{6}$$

$$(0 < x < a, \; 0 \leq y \leq b)$$

Since eq. (6) is not valid on the edges $x = 0$, $x = a$, further differentiation with respect to x cannot be accomplished term-by-term [12]. Assuming $w_{1,xxx}$ can be represented by a cosine-sine series, partial integration leads to the result

$$w_{1,xxx} = -\frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^3\pi^3}{a^3} A_{mn} + \alpha_m a_n + \beta_m b_n\right)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{7}$$

where

$$a_n = \frac{4}{ab}\int_0^b \left[w_{1,xx}(a,y) - w_{,xx}(0,y)\right]\sin\frac{n\pi y}{b}\,dy$$

$$b_n = \frac{4}{ab}\int_0^b \left[w_{1,xx}(a.y) + w_{,xx}(0,y)\right]\sin\frac{n\pi y}{b}\,dy$$

$$\alpha_m = 1,\ m \text{ even}$$
$$= 0,\ m \text{ odd}$$
$$\beta_m = 0,\ m \text{ even}$$
$$= 1,\ m \text{ odd}$$

A similar procedure applied to $w_{1,yy}$, $w_{2,x}$, $w_{2,y}$, $w_{2,xxx}$, and $w_{2,yyy}$ leads to ten more sets of constants: c_m, d_m associated with $w_{1,yy}$ along the edges $y = 0$, b; e_n, f_n associated with $w_{2,x}$ along the edges $x = 0$, a; g_m, h_m associated with $w_{2,y}$ along the edges $y = 0$, b; i_n, j_n associated with

$w_{2,xxx}$ along the edges $x = 0$, a; and k_m, l_m associated with $w_{2,yyy}$ along $y = 0$, b. All other desired derivatives can be obtained through term-by-term differentiation. Substituting the appropriate derivatives of w_1 and w_2 along with eq. (3) into eq. (1), equating like Fourier coefficients, and solving the resulting algebraic equations yields A_{mn}, B_{m0}, B_{0n}, and B_{mn} in terms of the coefficients a_n, b_n, c_m, . . ., l_m. The latter set of coefficients along with B_{00} are determined such that the boundary conditions are satisfied. Equation (1) also yields a relationship between i_0 and k_0.

For a plate having all edges simply-supported the following classical conditions, designated SS, are applicable.

at $x = 0$ and a:

$$w = M_x = - D_{11} w,_{xx} - 2D_{16} w,_{xy} - D_{12} w,_{yy} = 0 \tag{8}$$

at $y = 0$ and b:

$$w = M_y = - D_{12} w,_{xx} - 2D_{26} w,_{xy} - D_{22} w,_{yy} = 0 \tag{9}$$

The moment, or natural, boundary conditions lead to the following relationships

$$e_n = \frac{D_{11}b}{2D_{16}n\pi} a_n, \quad f_n = \frac{D_{11}b}{2D_{16}n\pi} b_n \qquad n \neq 0 \tag{10}$$

$$g_m = \frac{D_{22}Rb}{2D_{26}m\pi} c_m, \quad h_m = \frac{D_{22}Rb}{2D_{26}m\pi} d_m \qquad m \neq 0 \tag{11}$$

Thus eqs. (8) and (9) yield an infinite set of simultaneous equations for the constants B_{00}, a_n, b_n, c_m, d_m, e_0, f_0, g_0, h_0, i_n, j_n, k_m, and l_m. Truncation of the system at $m = M$ and $n = N$ leads to a $4(M + N + 2)$ system of equations. This system can be separated into four sets of equations by ranging m and n over even or odd integers. Any desired degree of convergence can be obtained by taking appropriate values of M and N.

For the purposes of this paper it is useful to also consider the combined boundary conditions, designated CS, where the edges x = 0, a are clamped and the edges y = 0, b are simply-supported. Thus, at x = 0 and b:

$$w = w,_{x} = 0 \tag{12}$$

and the conditions along the edges y = 0 and b are those of eq. (9).

The slope $w_{1,x}$ vanishes along the edges x = 0 and b if,

$$\sum_{m} \alpha_m \, m A_{mn} = 0, \quad \sum_{m} \beta_m \, m A_{mn} = 0 \tag{13}$$

for all n = 1, 2, ... ,

while $w_{2,x}$ vanishes along the same edges if

$$e_n = f_n = 0 \qquad (n = 0, 1, 2, \ldots,) \tag{14}$$

The previous system of eqs. for SS conditions can now be applied to the CS case by replacing eqs. (10) and the moment portion of eq. (8) with (14) and (13), respectively.

Using contracted notation the plate stresses and moment resultants are calculated form the constitutive relations

$$\sigma_i^{(k)} = z Q_{ij}^{(k)} \kappa_j$$

$$(i, j = 1, 2, 6) \qquad (15)$$

$$M_i = D_{ij} \kappa_j$$

where $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_6 = \sigma_{xy}$, and the curvatures, κ_i, and moments, M_i, are defined in an analogous manner. The curvature-displacement relations are those of classical homogeneous plate theory

$$\kappa_x = -w,_{xx}, \ \kappa_y = -w,_{yy}, \ \kappa_{xy} = -2w,_{xy} \qquad (16)$$

Buckling and Flexural Vibration

Consider the case of uniform membrane loading N_0, i.e.,

$$N_x = -K_1 N_0, \quad N_y = -K_2 N_0, \quad N_{xy} = K_3 N_0 \qquad (17)$$

$$N_0 \geq 0$$

where K_1, K_2 and K_3 are known constants. Solutions to eq. (1) with $q = 0$ are of the form

$$w = W(x, y)\, e^{i\omega t} \qquad (18)$$

where ω is the natural vibration frequency and t denotes time. Substituting eq. (18) into eq. (1) and taking eq. (17) into account yields

$$D_{11} W,_{xxxx} + 4D_{16} W,_{xxxy} + 2(D_{12} + 2D_{66}) W_{xxyy} + 4D_{26} W,_{xyyy}$$
$$+ D_{22} W,_{yyyy} - P\omega^2 W - K_1 N_0 W,_{xx} + 2K_3 N_0 W,_{xy} + K_3 N_0 W,_{yy} = 0 \qquad (19)$$

Solutions to eq. (19) can be obtained by assuming

$$W(x,y) = W_1(x,y) + W_2(x,y)$$

where W_1 and W_2 are of the same form as w_1 and w_2, respectively. Following the same procedure as in the previous section leads to four sets of homogeneous equations. Each set corresponds to a different mode shape. Natural vibration frequencies are obtained by letting $N_0 = 0$, while static buckling becomes the special case of vanishing ω. Proper values of N_0 and ω are determined by allowing the determinant of the coefficient matrix in each of the four groups of equations to vanish.

COMPARISON TO EXACT SOLUTION

An exact solution is available [17] for the bending deflection of a simply-supported anisotropic plate subjected to a uniformly distributed load q_0 and having the following elastic stiffness properties

$$\frac{Q_{11}}{Q_{22}} = 1, \frac{(Q_{12} + 2Q_{66})}{Q_{22}} = 1.5, \frac{Q_{16}}{Q_{22}} = \frac{Q_{26}}{Q_{22}} = -0.5 \tag{20}$$

This problem has been solved by the Fourier series method (FS) with $M = N = 1, 3, 5, 7, 9, 11$, and 13. Comparison of the maximum deflection for increasing terms with the exact solution (ES) is shown in Fig. 2. The maximum deflection is available from Reference [10] for the Ritz method (RS), and is also shown in Fig. 2 for $M = N = 1, 3, 5, 7$, and $M = 9$, $N = 8$. Since the Fourier series method satisfies all of the required boundary conditions, more rapid convergence is obtained compared to the energy approach in which only the geometric boundary conditions are satisfied. It should be noted, however, that direct comparison of the Fourier method and the Ritz method is difficult as given values of M and N lead to a different number of non-vanishing terms in the series representation of the deflection for the two approaches. Because of the symmetry of a uniform load, for example, $M = N = 7$ yields 25 non-zero terms in the Ritz method, while only 16 non-zero terms appear in the Fourier solution.

DISCUSSION

In the work by Ashton [10], it was shown that the rate of convergence for simply-supported anisotropic plates employing the Ritz method in conjunction with a double sine series was dependent on the anisotropy ratio Q'_{11}/Q'_{22}. Thus, for highly anisotropic materials such as graphite-epoxy the energy approach will provide a very slow converging solution for simple-support boundary conditions. This is illustrated in Fig. 3 for a single-ply composite subjected to a uniform transverse load and having the following properties with respect to the material symmetry axes

$$\frac{Q'_{11}}{Q'_{22}} = 25, \quad \frac{Q'_{12}}{Q'_{22}} = 0.25, \quad \frac{Q'_{66}}{Q'_{22}} = 0.5 \tag{21}$$

These are typical properties of a high modulus graphite-epoxy composite. The maximum deflection is shown for a square plate as a function of fiber orientation θ. Fourier series results are based on $M = N = 13$, while the Ritz solution is based on $M = N = 7$. In the case of the energy method this is the maximum number of terms available in the work of Ashton and Waddoups [6]. The difference between the Fourier solution and the energy solution is a maximum at orientations for which the D_{16} and D_{26} cross-elasticity terms have the greatest effect. Convergence of the energy method is immediately improved when two sides are clamped as also illustrated in Fig. 3. This improvement is due to the fact that the boundary conditions are all satisfied on the clamped side. For all sides rigidly clamped the Fourier method and the energy method both converge rapidly [14].

In Fig. 4, the bending moment across the centerline $y = b/2$ is shown for the 45^{o} oriented simply-supported plate of Fig. 3. It is interesting to note that the energy solution does not seem to be converging to the natural boundary condition at $x = 0$ and a. In particular, the bending moment M_x is increasing as it approaches the boundary rather than vanishing. Even though the proper bending deflections can be obtained without the natural boundary conditions being satisfied, the same is not necessarily true for functions such as bending moments which depend on partial derivatives of the deflection function. For the case of simple-supports it is very difficult to chose functions in conjunction with the Ritz method which will satisfy the natural boundary conditions. In particular, due to the existence of D_{16} and D_{26} cross-elasticity terms, the moment boundary conditions cannot be satisfied by a set of assumed functions in the separable form [18].

$$w = \sum_m \sum_n X_m(x)\, Y_n(y) \tag{22}$$

This conclusion is based on the assumption that all desired derivatives of eq. (22) can be obtained from term-by-term differentiation. It is also anticipated that similar difficulties would be encountered with finite element techniques which are based on principles of minimum potential energy.

The convergence of eigenvalve type problems is qualitatively similar to the maximum deflection results as illustrated by the buckling and vibration solutions in Tables I and II for a 45° oriented simply-supported square plate having the properties of eq. (21). Biaxial compression ($K_1 = K_2 = 1$, $K_3 = 0$) and pure shear ($K_1 = K_2 = 0$, $K_3 = 1$) are both shown.

In general, for a laminated plate of practical construction the magnitude of the D_{16} and D_{26} bending stiffness terms will decrease with an increase in the number of plies. Consider, for example, a symmetric angle-ply laminate ($+\theta, \ldots, -\theta, -\theta, \ldots, +\theta$ stacking sequence). The cross-elasticity bending stiffness terms are of the form

$$D_{16} = \frac{h^3\, Q_{16}(+\theta)}{4L}, \quad D_{26} = \frac{h^3\, Q_{26}(+\theta)}{4L} \tag{23}$$

while the remaining bending stiffness terms are independent of the number of plies. Thus, the effect of cross-elasticity bending stiffness terms is less severe in a laminated plate. This is illustrated in Table III where the deflection and bending moment, M_x, at the center of a 4 layer simply-supported $\pm 45^{\circ}$ ($+45^{\circ}$, -45°, -45°, $+45^{\circ}$) laminate are tabulated for increasing values of M and N. Unidirectional ply properties are those of eq. (21). The difference between the Fourier solution and the Ritz solution for plate deflection is much less than in the case of a single layer $+45^\circ$ plate. It should be noted, however, that the bending moment as determined by the Ritz method displays the same erratic behavior previously noted by Ashton [10] for a homogeneous anisotropic plate. In particular, the bending moment at the center seems to be oscillating about a value close to the results given by the Fourier solution for $M = N = 13$. Furthermore, the amplitude of the oscillation seems to become larger with increasing values of M and N, strongly suggesting that M_x does not converge. Further evidence suggesting such a conclusion is illustrated in Figure 5. The distribution of M_x across the center of the plate according to the Ritz solution for $M = N = 3$ does not look too unreasonable compared to the Fourier solution. For $M = N = 7$, however, the Ritz method yields results which are rather horrifying.

It should also be noted that the Fourier solution yields an oscillating convergence for M_x at the center of the plate. A cursory examination of Table III, however, reveals that the oscillations damp out rapidly with increasing values of M and N.

CONCLUSIONS

Numerical results show that a classical Fourier analysis can yield improved solutions for homogeneous and symmetrically laminated anisotropic plates having strong cross-elasticity bending stiffness terms compared to existing solutions obtained by energy methods. This improvement is due to the fact that the Fourier solution satisfies both the geometric and natural boundary conditions, while the energy method in conjunction with classical beam mode functions satisfies only the geometric boundary conditions. Convergence of the Ritz solutions for deflections, buckling loads, and vibration frequencies seem to be quite slow for highly anisotropic plates in which all four edges involve natural boundary conditions. It is quite questionable, however, whether functions, such as bending moments, which involve derivatives of the plate deflection converge at all.

The Fourier series solution appears to provide a conservative solution for the results presented. In particular, bending deflections are larger than the exact solution, while buckling loads and fundamental vibration frequencies are low compared to the energy solution which is an upper bound.

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TABLE I

Buckling of 45° Simply-Supported Plate, a = b

$N_0 b^2/Q'_{22} h^3$ - Biaxial Compression ($K_1 = K_2 = 1$, $K_3 = 0$)			
M	N	Fourier Analysis	Reference 7
1	1	6.763	21.438
3	3	8.115	13.013
5	5	8.318	11.565
7	7	8.418	11.060
9	9	8.481	------
11	11	8.521	------
13	13	8.556	------
$N_0 b^2/Q'_{22} h^3$ - Uniform Shear ($K_1 = K_2 = 0$, $K_3 = 1$)			
1	1	12.121	∞
3	3	13.309	27.953
5	5	13.567	18.022
7	7	13.702	17.122
9	9	13.786	------
11	11	13.843	------
13	13	13.884	------

TABLE II

Fundamental Vibration Frequency of 45°

Simply-Supported Plate, a = b

$\omega b^2 (\rho / Q'_{22} h^3)^{1/2}$			
M	N	Fourier Analysis	Reference 7
1	1	13.277	20.571
3	3	14.231	17.301
5	5	14.415	16.737
7	7	14.496	16.428
9	9	14.544	------
11	11	14.577	------
13	13	14.600	------

TABLE III

Bending of Simply-Supported ±45° Laminate

(+45°, -45°, -45°, +45°), a = b, q = q_o

		$w(a/2, b/2)\, Q_{22} h^3 / q_o b^4 \times 10^3$		$M_x(a/2, b/2) / q_o b^2 \times 10^2$	
M	N	F S	R S	F S	R S
1	1	5.1838	3.8310	4.7785	4.1854
3	3	4.9676	4.3850	4.0025	4.2457
5	5	4.8849	4.3947	4.2311	3.6911
7	7	4.8571	4.4910	4.1488	4.2633
9	9	4.8307	------	4.1716	------
11	11	4.8193	------	4.1533	------
13	13	4.8078	------	4.1596	------

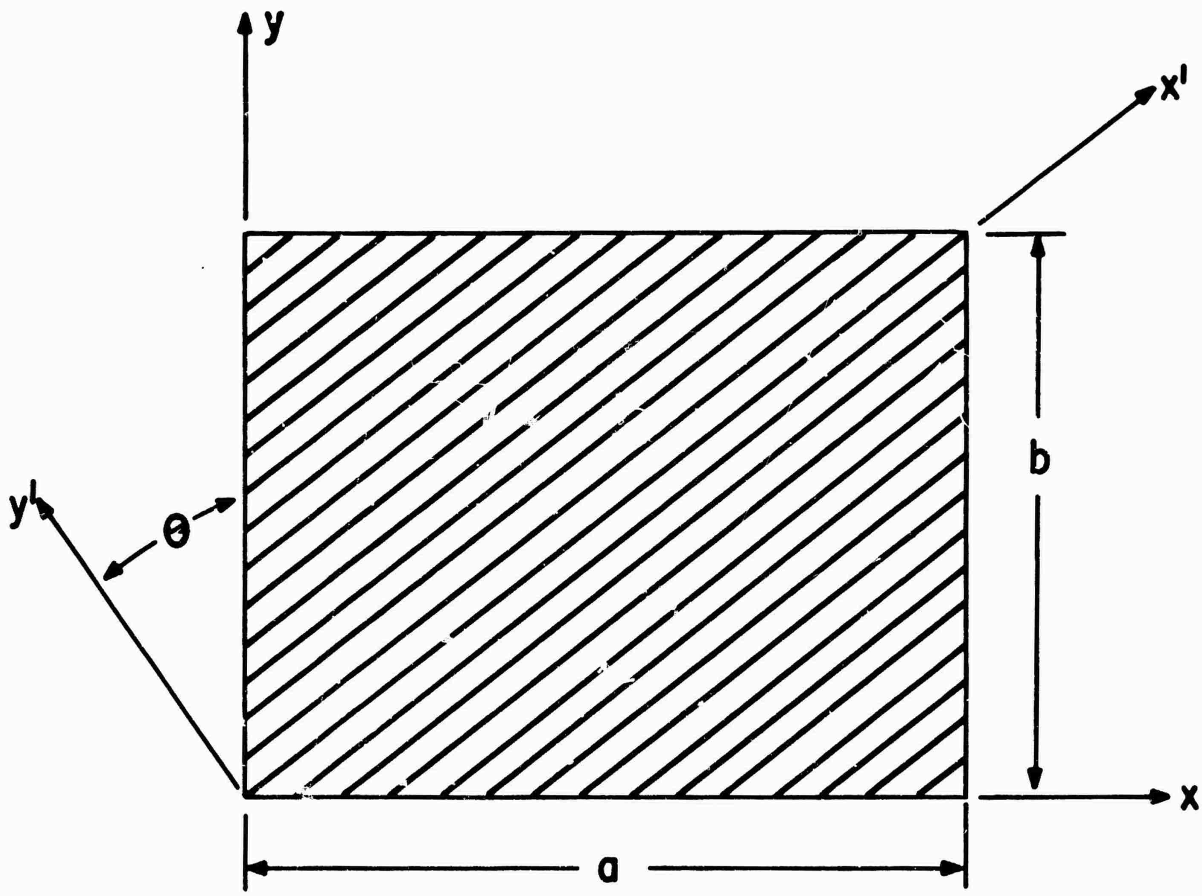

FIGURE 1. Coordinates

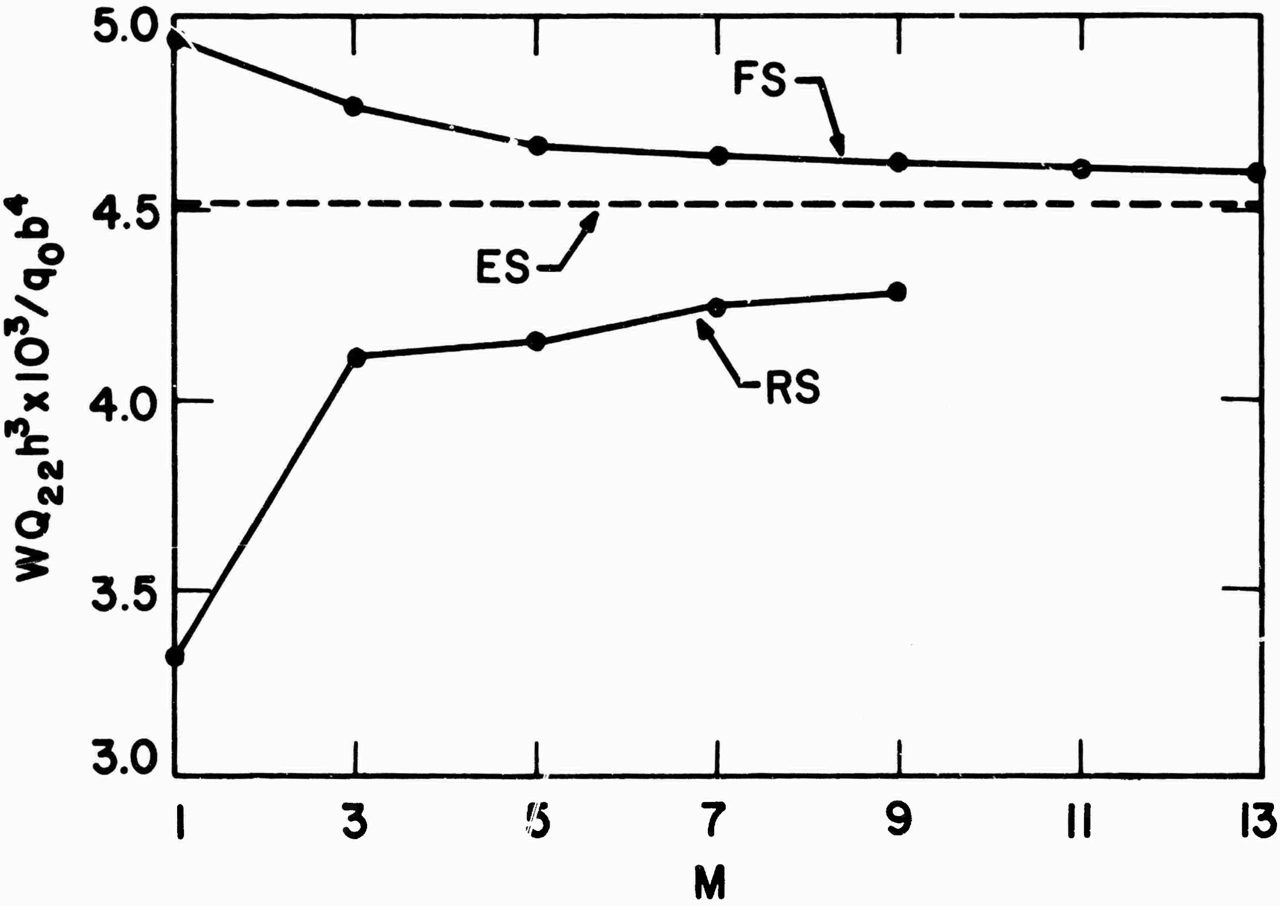

FIGURE 2. Maximum Deflection for Increasing Number of Terms

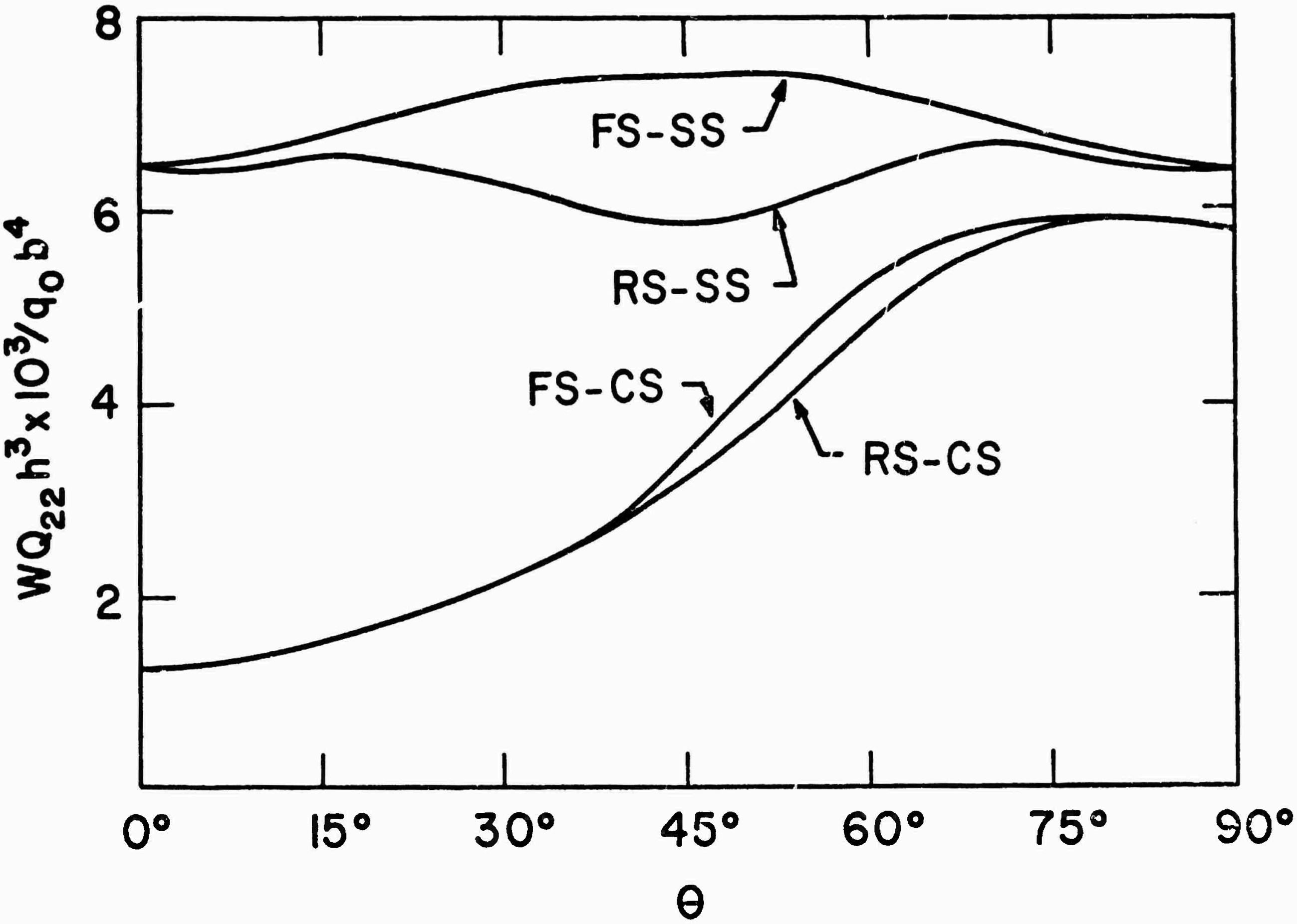

FIGURE 3. Deflection vs. Orientation, Uniformly Loaded Plate, a = b

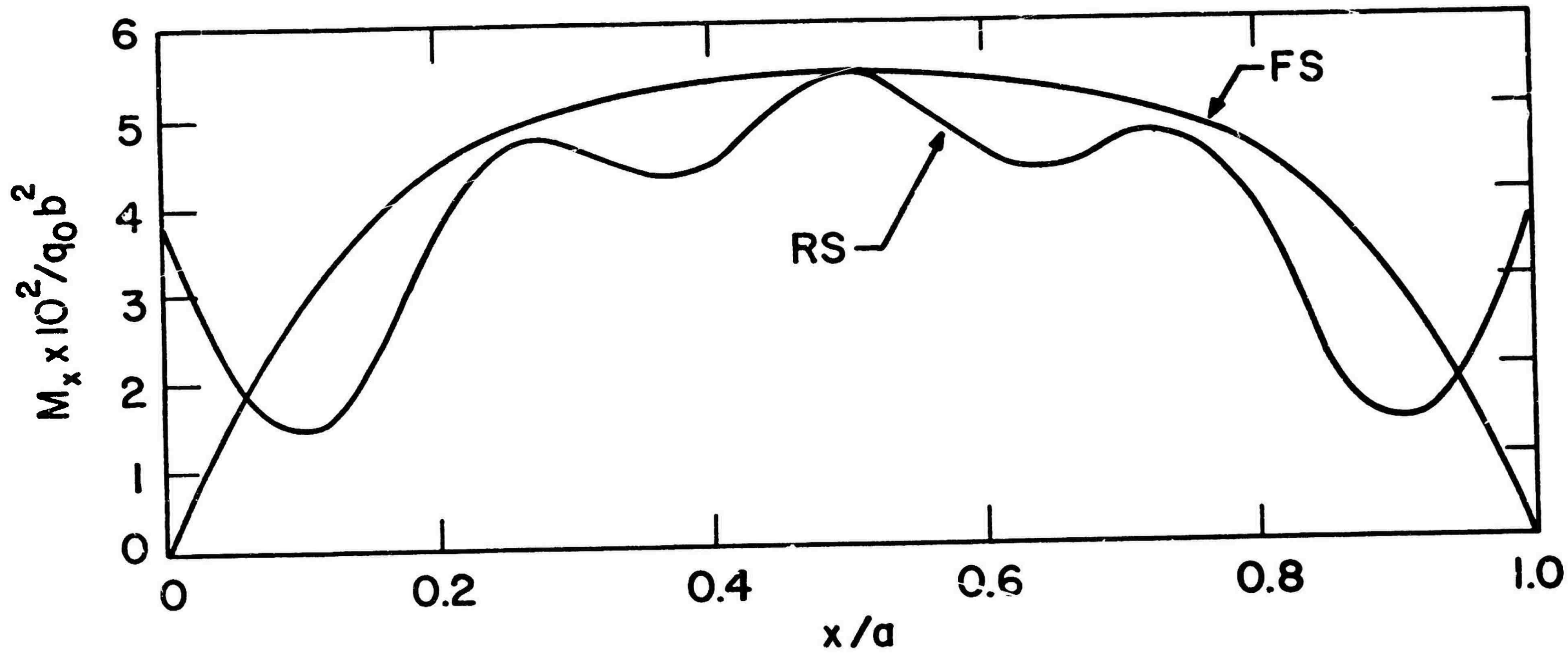

FIGURE 4. Bending Moment $M_x(x, b/2)$, Simply-Supported Plate Under Uniform Load, $a = b$

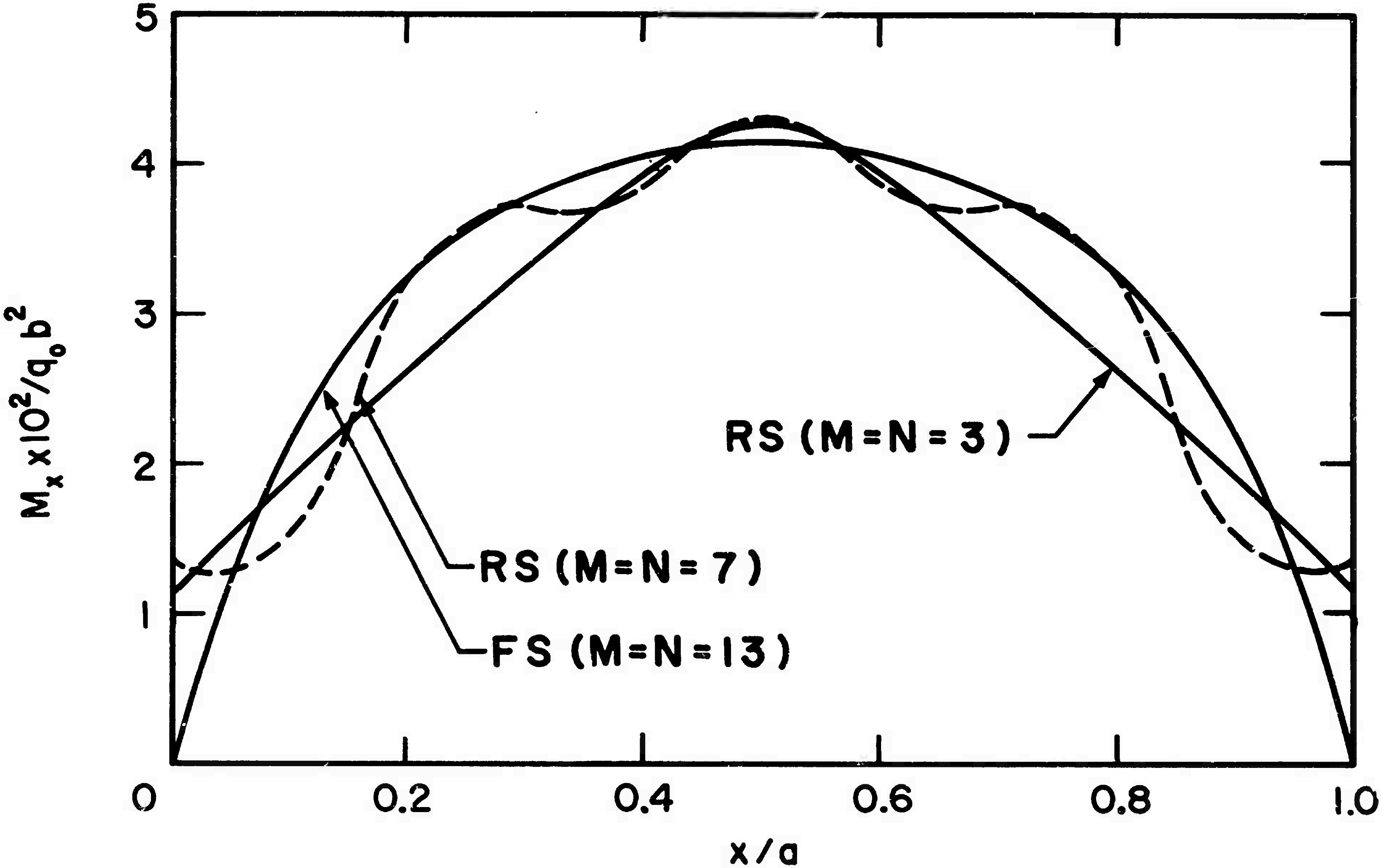

FIGURE 5. Bending Moment $M_x(x, b/2)$, Simply-Supported Laminate (+45°, -45°, -45°, +45°) Under Uniform Lead, a = b